Overview of different upscaling regimes for wave propagation in random media

experimental observations, numerical experiments, analysis techniques

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Outline

Wave propagation in a heterogeneous medium

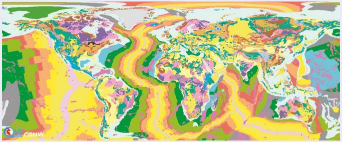
Elastic wave equation in a heterogeneous medium

$$\rho(\boldsymbol{x}) \frac{\partial^2}{\partial t^2} \boldsymbol{u}(\boldsymbol{x}, t) = \nabla \cdot (\mathrm{C}(\boldsymbol{x}) : \nabla \otimes \boldsymbol{u}(\boldsymbol{x}, t)) + \boldsymbol{f}(\boldsymbol{x}, t)$$

- The material density $\rho(\boldsymbol{x})$ and constitutive tensor $\mathrm{C}(\boldsymbol{x})$ fluctuate over a characteristic size ℓ_c.
- The wave field is characterized by its wavelength λ.
- The source-observation distance is denoted L.
- The amplitude of fluctuations is denoted σ.

Wave propagation in a heterogeneous medium : geophysics

Examples of heterogeneities



(a) at the global scale[1] ($\ell_c \approx 1000$ km)

(b) at the local scale ($\ell_c \approx 1$ km)

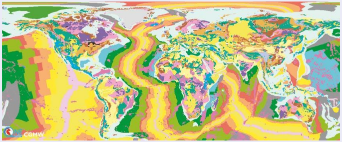
1. K. Charles. "Mother Earth gets undressed". In : *Nature News* (2008). DOI : 10.1038/news.2008.1001

2. R.-S. Wu et K. Aki. "Introduction : seismic wave scattering in three-dimensionally heterogeneous Earth". In : *Pure Appl. Geophys.* (1988). Pageoph Topical Volume "Scattering and Attenuations of Seismic Waves", p. 1-6. DOI : 10.1007/978-3-0348-7722-0_1

3. H. Sato, M. C. Fehler et T. Maeda. *Seismic wave propagation and scattering in the heterogeneous earth*. Second Edition. Springer, 2012

Wave propagation in a heterogeneous medium : geophysics

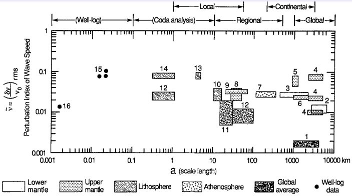
Examples of heterogeneities



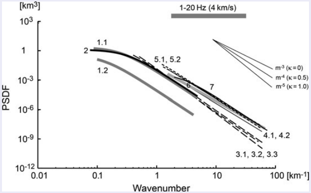
(a) at the global scale [1] ($\ell_c \approx 1000$ km)

(b) at the local scale ($\ell_c \approx 1$ km)

Distribution of fluctuations



(a) Distribution of heterogeneities in the Earth [2]



(b) Experim. PSDF of fractional velocity [3]

1. K. Charles. “Mother Earth gets undressed”. In : *Nature News* (2008). DOI : 10.1038/news.2008.1001

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3. H. Sato, M. C. Fehler et T. Maeda. *Seismic wave propagation and scattering in the heterogeneous earth*. Second Edition. Springer, 2012

Correlation between properties [4]

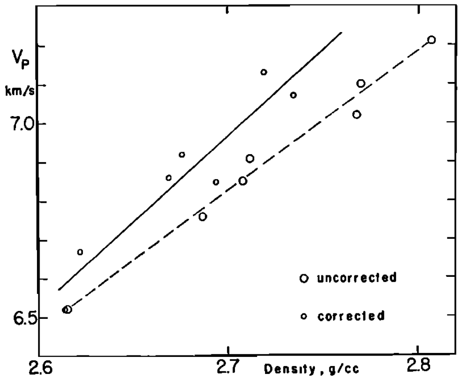
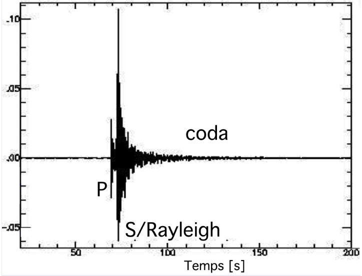


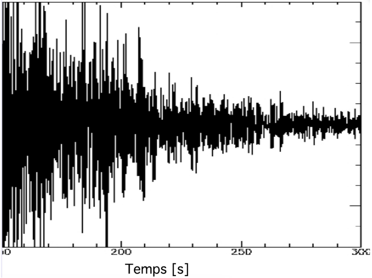
FIGURE – Velocity at 10 kilobars versus density for rocks composed mainly of plagioclase feldspar.

4. F. BIRCH. "The velocity of compressional waves in rocks to 10 kilobars : part 2.". In : *J. Geophys. Res.* 66.7 (1961), p. 2199-2224. DOI : 10.1029/JZ066i007p02199

Typical observation : surface accelerograms



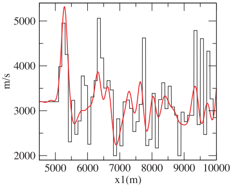
(a) Full wave field



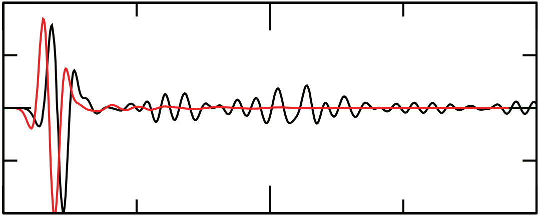
(b) zoom on the coda

- Few well-marked pulses (volume and surface waves) are followed by a seemingly random coda

Characteristics of coherent pulses versus coda : average medium [5]



(a) Shear velocity map (black)



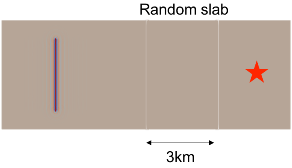
(b) Solution in average-velocity medium (red) versus true (black)

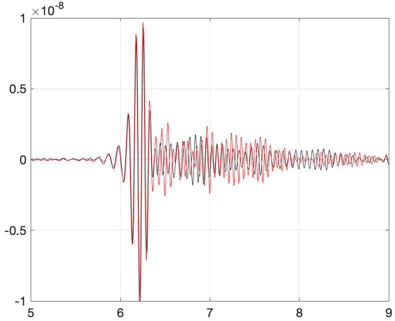
Propagation in a medium with average velocity does not reproduce the feature of the true medium :

- The coherent pulse is too fast
- The coda is lacking

5. Y. Capdeville, L. Guillot et J.-J. Marigo. "2D non-periodic homogenization to upscale elastic media for P-SV waves". In : *Geophys. J. Int.* 182.2 (2010), p. 903-922. DOI : 10.1111/j.1365-246X.2010.04636.x

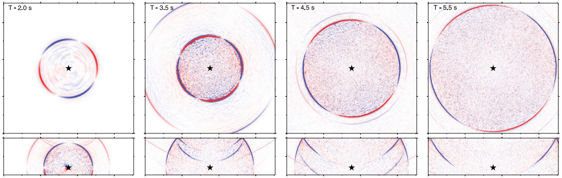
Characteristics of coherent pulses versus coda : stability





- Coherent pulses are less sensitive than coda to the precise location of heterogeneities

Characteristics of coherent pulses versus coda : anisotropy of wave field [6]

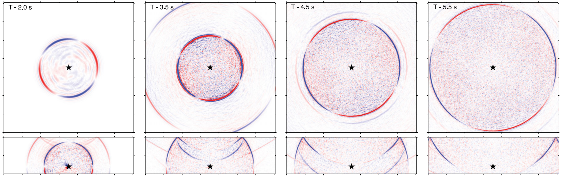


(a) Weak interaction $\lambda \gg \ell_c$

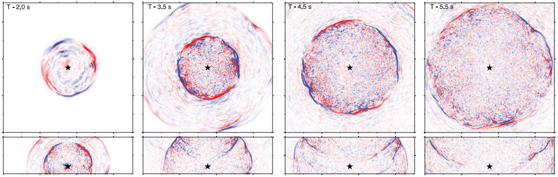
- Coda sees less directionality effects than first arrivals

6. W. Imperatori et P. M. Mai. "Broad-band near-field ground motion simulations in 3-dimensional scattering media". In : *Geophys. J. Int.* 192.2 (2013), p. 725-744. DOI : 10.1093/gji/ggs041

Characteristics of coherent pulses versus coda : anisotropy of wave field [6]



(a) Weak interaction $\lambda \gg \ell_c$

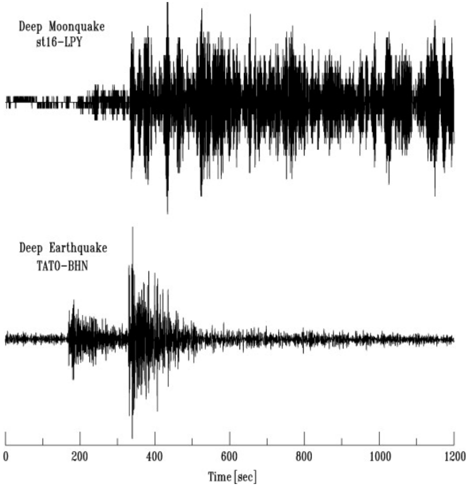


(b) Strong interaction $\lambda \approx \ell_c$

- Coda sees less directionality effects than first arrivals
- Influence of λ/ℓ_c on relative amplitude of pulses and coda

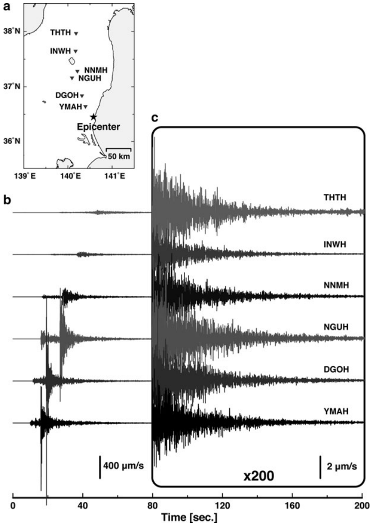
6. W. Imperatori et P. M. Mai. "Broad-band near-field ground motion simulations in 3-dimensional scattering media". In : *Geophys. J. Int.* 192.2 (2013), p. 725-744. DOI : 10.1093/gji/ggs041

Characteristics of coherent pulses versus coda : relative amplitudes [7]



7. C. Frohlich et Y. Nakamura. "The physical mechanisms of deep moonquakes and intermediate-depth earthquakes : how similar and how different ?" In : *Phys. Earth Planetary Interiors* 173 (2009), p. 365-374. DOI : 10.1016/j.pepi.2009.02.004

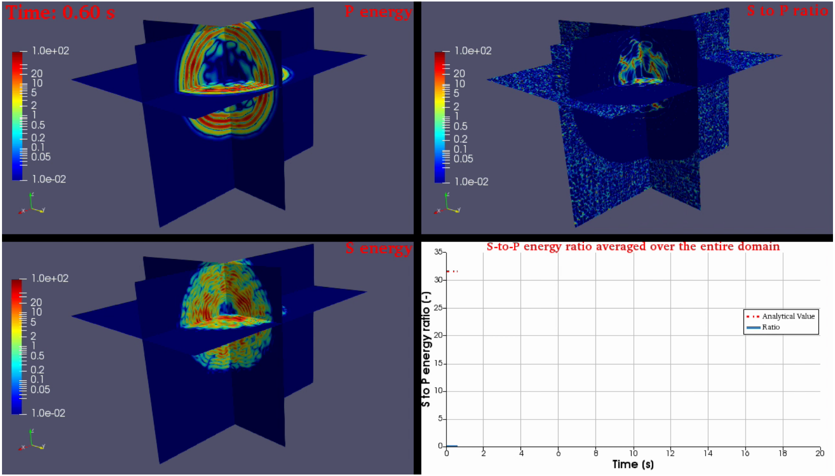
Characteristics of the late coda : influence of distance[8]



- Coda amplitude at long times is less dependent on distance to source L than first arrivals

8. H. Sato, M. C. Fehler et T. Maeda. *Seismic wave propagation and scattering in the heterogeneous earth*. Second Edition. Springer, 2012.

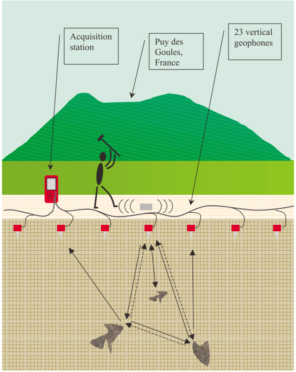
Characteristics of the late coda : equipartition [9]



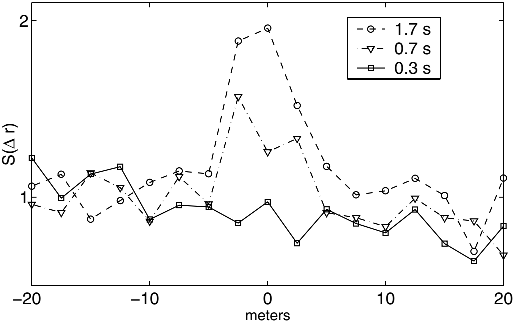
- Energy tends to spread uniformly over all possible modes

9. S. Khazaie et R. Cottereau. "Influence of local cubic anisotropy on the transition towards an equipartition regime in a 3D texture-less random elastic medium". In : *Wave Motion* 96.102574 (2020), p. 1-18. DOI : 10.1016/j.wavemoti.2020.102574

Weak localization (coherent backscattering) [10]



(a) Experimental setup



(b) Evolution of (normalized) energy in space

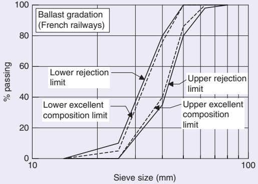
- Part of the energy of the coda returns and localizes close to the source position

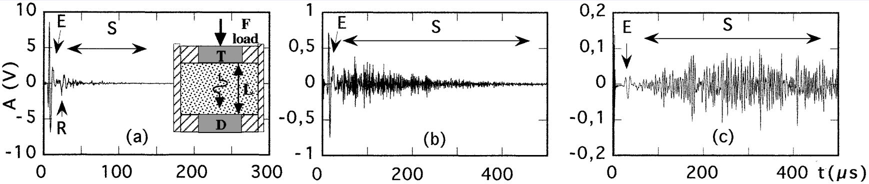
10. E. Larose et al. "Weak localization of seismic waves". In : *Phys. Rev. Lett.* 93.4 (2004), p. 048501. doi : 10.1103/PhysRevLett.93.048501

Wave propagation in granular media

Vibrations in a ballasted railway track [11]

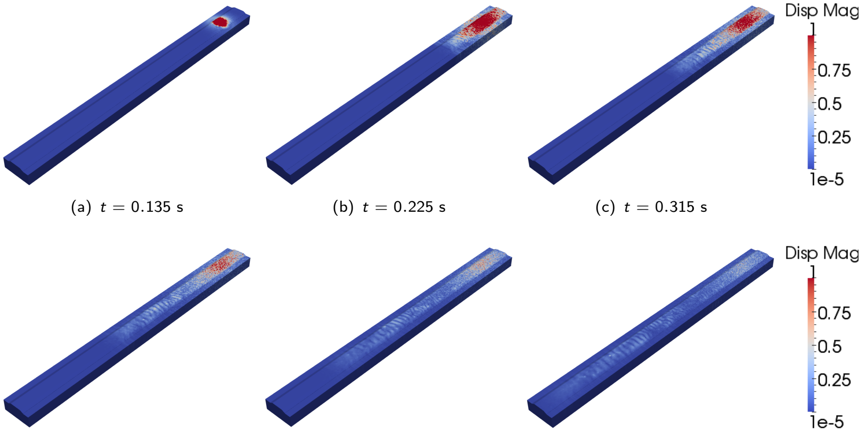
Pressure waves in granular media ($\ell_c \approx 10$ cm)





11. X. Jia, C. Caroli et B. Velicky. "Ultrasound propagation in exernally stressed granular media". In : *Phys. Rev. Lett.* 82.9 (1999), p. 1863-1866. DOI : 10.1103/PhysRevLett.82.1863
B. Indraratna, W. Salim et C. Rujikiatkamjorn. *Advanced rail geotechnology. Ballasted track*. CRC Press, 2011

Strong (Anderson) localization [12]



(a) $t = 0.135$ s (b) $t = 0.225$ s (c) $t = 0.315$ s

(d) $t = 0.405$ s (e) $t = 0.495$ s (f) $t = 0.585$ s

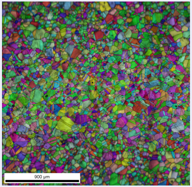
FIGURE – Normalized displacement fields in the ballasted railway track.

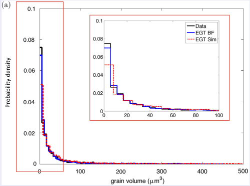
12. L. de Abreu Corrêa, R. Cottereau et B. Faure. "Dispersion analysis in ballasted railway tracks and Anderson localization in granular media". In : *J. Sound Vib.* 465.115010 (2019), p. 1-13. doi : 10.1016/j.jsv.2019.115010

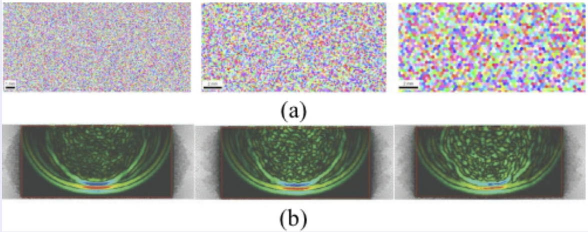
Wave propagation in polycrystalline materials

Damage in polycrystalline materials [13]

Ultrasonic waves in polycrystalline materials ($\ell_c \approx 10\ \mu$m)







13. **Tie2010**

P. Mu. "Study of crack initiation in low-cycle fatigue of an austenitic stainless steel". Thèse de doct. France : École Centrale de Lille, 2011

K. Teferra et L. Graham-Brady. "Tesselation growth models for polycrystalline microstructures". In : Comp. Mat. Sci. 102 (2015), p. 57-67. doi : 10.1016/j.commatsci.2015.02.006

Summary of general observations

- There are two phases with very different features in recordings : the first pulses (coherent) and the coda (incoherent)

The coherent pulses

- Seem deterministic with an amplitude strongly dependent on distance to source L
- Have strong directionality/anisotropy features
- Are not sensitive to the particular realization of heterogeneity
- Are stronger (relatively to coda) when weak heterogeneities fluctuate faster than wavelength $\lambda \gg \ell_c$ and $\sigma \ll 1$

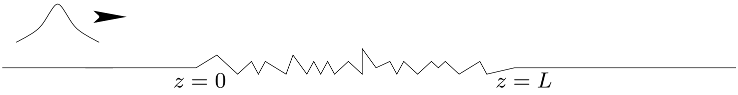
The coda

- Seems random with an amplitude independent (at late times) on L
- Seems to propagate isotropically
- Is sensitive to the particular realization of heterogeneity
- Is stronger when $\lambda \approx \ell_c$ and $\sigma \approx 1$

- Homogenized models should be able to reproduce these features, random and deterministic

Outline

Identification of scaling regimes [14]



Let us consider the 1D heterogeneous acoustic wave equation in a slab in between two homogeneous half-spaces

$$\begin{aligned} \rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} &= F(t,z) \\ \frac{1}{K(z)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} &= 0 \end{aligned}$$

14. J.-P. Fouque et al. *Wave propagation and time reversal in randomly layered media*. T. 56. Stochastic Modelling and Applied Probability. Springer, 2007.

Modeling of the random medium

The bulk modulus is modeled as a mean-zero stationary random process ν_K

$$\rho(z) = \overline{\rho}, \quad \frac{1}{K(z)} = \begin{cases} \frac{1}{\overline{K}} & \text{if } z < 0 \\ \frac{1}{\overline{K}}\left(1 + \epsilon\nu_K(z)\right) & \text{if } z \in [0, L] \\ \frac{1}{\overline{K}} & \text{if } z > L \end{cases}.$$

with standard deviation and correlation length

$$\sigma^2 = \mathbb{E}\left[\nu_K(z_0)^2\right], \quad \sigma^2 \ell_c = \int_{\mathbb{R}} \mathbb{E}\left[\nu_K(z_0)\nu_K(z_0 + z)\right] dz.$$

Finally, the random process is rescaled

$$\nu_K(z) = \sigma\nu\left(\frac{z}{\ell_c}\right)$$

Modeling of the source term

We introduce a point source located in the left homogeneous half-space at some $z_0 < 0$

$$F(t,z) = \bar{\zeta}^{1/2} g(t)\delta(z - z_0)$$

We define the pulse width

$$T_0^2 = \frac{\int_{\mathbb{R}} (t - \overline{T})^2 g^2(t)dt}{\int_{\mathbb{R}} g^2(t)dt}, \quad \text{where,} \quad \overline{T} = \frac{\int_{\mathbb{R}} t g^2(t)dt}{\int_{\mathbb{R}} g^2(t)dt}$$

and the typical frequency $\omega_0 = 2\pi/T_0$, so that we use the normalized pulse shape

$$g(t) = f(\omega_0 t)$$

NB : in our applications, the carrier frequency and the bandwidth are of the same order of magnitude : ω_0

The dimensionless wave equation

We finally rescale time and space as

$$\tilde{z} = \frac{z}{L}, \quad \tilde{t} = \frac{\overline{c}t}{L}$$

where L is the typical wave propagation distance and $\overline{c}^2 = \overline{K}/\overline{\rho}$ is a reference speed of propagation. Introducing also $\overline{\zeta} = \overline{\rho c}$, we obtain the normalized pressure and velocity fields

$$\tilde{p}(\tilde{t}, \tilde{z}) = \overline{\zeta}^{-1/2} p\left(\tilde{t}\frac{L}{\overline{c}}, \tilde{z}L\right), \quad \tilde{u}(\tilde{t}, \tilde{z}) = \overline{\zeta}^{1/2} u\left(\tilde{t}\frac{L}{\overline{c}}, \tilde{z}L\right)$$

and, using the dimensionless wave equation

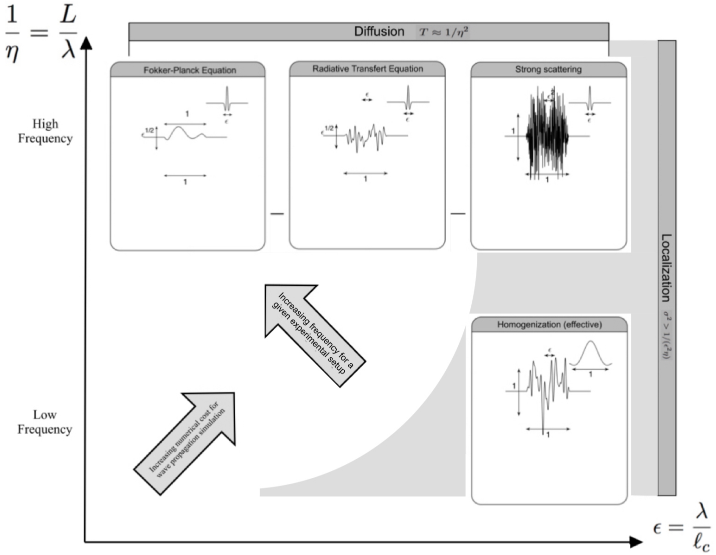
$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{z}} &= f\left(\tilde{t}\frac{\omega_0 L}{\overline{c}}\right)\delta(\tilde{z} - \tilde{z}_0) \\ \left(1 + \sigma\nu\left(\tilde{z}\frac{L}{\ell_c}\right)\right)\frac{\partial \tilde{p}}{\partial \tilde{t}} + \frac{\partial \tilde{u}}{\partial \tilde{z}} &= 0 \end{aligned}$$

Three independent dimensionless group of parameters appear

1. The strength of fluctuations σ
2. Two length ratios : L/ℓ_c and $\omega_0 L/\overline{c} \approx L/\lambda$.

Dynamic homogenization : what happens to the wave field in the different dynamical regimes (defined by the relative values of the above coefficients)

The scaling regimes



The scaling regimes

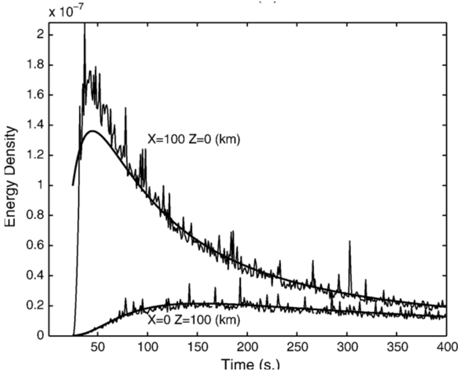


FIGURE – Energy density computed from wave equation and diffusion [15]

- Homogenized models provide upscaled models explaining the impact of small-scale phenomena at the larger scales.
- Some homogenized models are stochastic, some are deterministic (sometimes both are available in the same regime).

15. L. MARGERIN. "Attenuation, transport and diffusion of scalar waves in textured random media". In : *Tectonophys.* 416.1-4 (2006), p. 229-244. DOI : 10.1016/j.tecto.2005.11.011

Overview of different upscaling regimes for wave propagation in random media

experimental observations, numerical experiments, analysis techniques

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